State-Space Estimation of Multi-Factor Models of the Term Structure:

Application to Government of Jamaica Bonds

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Abstract

The econometric estimation of the term structure of interest rates has received tremendous attention from financial- and macro-economists. Measurement of the term structure of interest rates allows for the extraction of information on investor's expectations about future interest rates. This paper estimates two famous equilibrium models of the term structure of interest rates using zero-coupon Government of Jamaica (GOJ) bonds. Multi-factor versions of the Vasicek (1977) and the Cox, Ingersoll and Ross (CIR; 1985) models of the nominal interest rate term structure are estimated using a state-space approach. This approach simultaneously integrates times series and cross-sectional GOJ bond yields to generate the unobservable state variables using a Kalman filter. One to three factor models are estimated using a quasi-maximum-likelihood technique. Statistical tests confirm that the two-factor CIR-models best accounts for the dynamics of the term structure. The results indicate that the extracted factors are closely related with the general level and 'steepness' of interest rates. The empirical analysis for the 2-factor model revealed that the level of the short rate exhibited strong and smooth mean reversion and indicated the existence of a large and significant risk premium that increases with time to maturity. The study finds evidence that supports the usefulness of term structure models for monetary policy. Based on estimated factor loadings, this study concludes that the unobserved short rate (related to the BOJ policy rate) has a significant impact on the short end of the yield curve but a relatively minimal impact on the long end.

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1.0 Introduction

The econometric estimation of the term structure of interest rates has received tremendous attention from financial- and macro-economists, particularly in the context of bond pricing (see, for example, Babbs and Nowman (1999), Dai and Singleton (2000) and Pearson and Sun (1994)). Based on the Expectations Theory of the term structure, the yields on long-term bonds are the expected value of risk-adjusted average future short-term yields. Hence, measurement of the term structure of interest rates allows for the extraction of information on investors' expectations about future interest rates. Term structure measurement models have a range of applications. Specifically, interpreting the empirical properties of bond yield dynamics that are provided by term structure measurement models is important for a number of purposes that include:

- Influencing aggregate demand through monetary policy (see, for example, Ang and Piazzesi (2003), Diebold, Redebusch and Aruoba (2003), Fendel (2004), Hördahl, Tristani and Vestin (2002), Piazzesi (2003) and Rudebusch and Wu (2003)). The short rate is the fundamental policy instrument of the central bank. That is, central banks may shift the short end of the yield curve when adjusting their policy stance. However, movement in long-term rates have a greater influence on aggregate demand. Thus, knowledge of yield curve dynamics provides information to the central bank on how their interest rate decisions will impact the future path of the economy.
- Risk management through the pricing and hedging of interest rate-contingent claims including caps, floors and swaptions (see, for example, Amin and Morton (1994), Buhler *et al* (1999), Driessen, Klaasen and Melenberg (2002), Canabarro (1995), Chernov and Ghysels (2000), Jagannathan *et al* (2000) and Longstaff *et al* (2001). Further, value-at-risk estimates for fixed income portfolios can be obtained through simulating paths for the term structure.[1]
- Public debt management through bond issues (see, for example, Dai and Philippon (2004)). Knowledge of the dynamic properties of the yield curve provides information on the impact of fiscal policy on investor risk preferences and future yield expectations of bonds across maturities. Fiscal authorities can use this information when deciding the length of tenors in their financing decisions.

[1] Value-at-Risk is defined as the maximum potential loss on a portfolio for a given horizon and probability.

Vasicek (1977) introduced the first no-arbitrage partial equilibrium model of the term structure in which the evolution of the short rate exhibits mean reversion. The no-arbitrage condition is necessary to ensure that the future evolution of bond prices of different maturities are correctly priced and do not allow for arbitrage opportunities. That is, the market price of risk, which is assumed to be an exogenous parameter, should be the same across different asset maturities. The mean reversion requirement stipulates that the short rate will tend to drift back to an underlying rate. The conditional mean and variance of the stochastic process for the short-term interest rate can be computed allowing for the derivation of a closed form solution. However, a drawback of the Vasicek model is that the assumption of a Gaussian interest rate process allows for negative short rates.

The CIR (1985) model is the first general equilibrium model to explain the term structure of interest rates in a well-defined economic environment. The CIR model is developed under the assumptions of infinitely lived and single-good economy with identical consumers who maximize a time-additive utility function with logarithmic preferences. The single good is produced stochastically with a linear production function that evolves in continuous time with the expected return vector and covariance matrix dependent on the stochastic evolution of a vector of state variables. Equilibrium is achieved when the wealth of individuals is totally invested in the firms based on the choice of consumption and investment allocations that maximise their expected utility. The main difference between the Vasicek and CIR models is that the short rate is specified as a square root process that is proportional to the level of the interest rate. This feature prevents the occurrence of negative rates under certain restrictions, unlike the Vasicek case. Additionally, the market price of risk is determined endogenously which facilitates the internal consistency of the model.[2]

[2] See Subrahmanyam (1996) for a detailed discussion on the Vasicek and CIR models as well as other seminal term structure models.

Term structure models were originally estimated with either time series bond yields or a cross-section of bond yield over different maturities. The time series approach incorporates the intertemporal dynamics of the term structure but not cross-section information.[3] However, to ensure the model is arbitrage free, a range of maturities should be included in the estimation. The cross-section approach which uses bond yield data across maturities at a point in time has the drawback that the parameters can be unstable over different points in time.[4] Hence, the incorporation of both time series and cross-section data in empirical tests of the term structure allows for the proper use of information from both dimensions in order to obtain more accurate parameter estimates.[5] Nevertheless, a main drawback of time series/cross-section models of the term structure is that if the number of maturities is larger than the number of factors, the model will be under identified. In order to circumvent this problem, most term structure models that rely on panel data add Gaussian measurement errors when estimating the relationship between the maturity yields and the unobserved state factors to obtain consistent parameters. The inclusion of measurement errors is consistent with the existence of market regularities such as bid-ask spreads and non-synchronous trading.

Single-factor term-structure models describe the dynamics of the instantaneous short rate. Hence, these models can only account for parallel shifts in the yield curve. In practice, however, other factors may influence different sections of the yield curve allowing for various shapes such as twists and inverse humps. Alternatively, the flexibility inherent in multi-factor term-structure models allows for a wider range of possible yield curve shapes. Three-factor term-structure models are usually estimated in practice to explain the dynamics of the term structure of interest rates. The specification of three factors rely on the seminal study by Litterman and Scheinkman (1991), based on standard principle component analysis,

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[3] Examples of recent term structure models that rely on time series data include: Anderson and Lund (1997), Brenner, Harjes and Kroner (1996), Broze, Scaillet and Zakoian (1995) and Chan, Karolyi, Longstaff and Sanders (1992).
[4] Examples of recent term structure models that rely on cross-section data include: Brown and Dybvig (1986), Brown and Schaefer (1994) and De Munnik and Shotman (1994).
[5] Examples of recent term structure models that incorporate both times series and cross-section data include: Babbs and Nowman (1999), Ball and Torous (1996), Chatterjee (2005), Chen and Scott (1995), De Jong (2000), Duan and Simonato (1995), Geyer and Pichler (1996), Gibbons and Ramaswamy (1993), Jegadeesh and Pennacchi (1996), Lund (1997), Pearson and Sun (1994), Pennacchi (1991).

which found that three factors corresponding to the level, curvature and slope of the yield curve explained the term structure of US Treasury bond yields in the 1980s. However, many studies have found that the inclusion of additional factors does not increase the performance of term structure models.[6] Consistent with this finding, the Litterman and Scheinkman (1991) study determined that almost 90.0 per cent of the variation in US Treasury yields was driven by the variation in the first factor.

This paper estimates two famous equilibrium models of the term structure of interest rates using zero-coupon Government of Jamaica (GOJ) sovereign bonds for the period 24 September 2004 to 28 July 2006. Specifically, multi-factor versions of the Vasicek (1977) and the Cox, Ingersoll and Ross (CIR; 1985) models of the nominal interest rate term structure are estimated using a state-space approach. This approach simultaneously integrates time series and cross-sectional GOJ sovereign yields to generate the unobservable state variables using a Kalman filter. The objective of this exercise is to explain the yield curve dynamics in Jamaica in order to derive information on investor expectations to support monetary and fiscal policy objectives as well as to accurately price bonds and hedging instruments.

The next section focuses on the continuous-time formulation of term structure models beginning with the preliminaries associated with term structure modelling in continuous-time. The state space representation of the Vasicek and CIR term structure models and the Kalman filter algorithm will then be presented in Section 3. One to three factor versions of these models will be used to explain the dynamics of the term structure of GOJ bonds for the period 24 September 2004 to 28 July 2006. The data description and empirical results are reported in Section 4. Section 5 provides a brief conclusion and policy recommendations.

[6] See, for example, Chatterjee (2005).

2.0 Equilibrium Models of the Term Structure

The Vasicek (1977) and CIR (1985) models fall in the class known as "equilibrium models of the term structure." These models rely on specific assumptions about the stochastic nature of state variables to obtain information on the dynamic evolution of the term structure within an economic environment. The distinct features of these models are that the *market price of risk* is identified either exogenously or endogenously and the instantaneous short rate is explicitly specified as a function of unobserved state variables. The market price of risk, otherwise called the *Sharpe ratio*, refers to the expected standardised excess rate of return above the risk free rate from a specific zero-coupon bond.

The Kalman filter is a relatively new econometric technique that has been used in the recent finance literature to estimate the continuous-time multi-factor Vasicek and CIR affine term-structure models.[7] This method is used to determine the relationship between market bond yields and the unobserved state variables that drive them.[8] Kalman filter estimation involves the estimation of two systems of equations known as state-space estimation. The measurement system specifies the affine relationship between an observed set of zero-coupon bond yields and unobserved state variables. The system of transition equations models the dynamics of the state variables. The Kalman filter algorithm recursively formulates an optimal predictor of the unobservable state variable vector from the system of transition equations conditional on the measurement system observed zero-coupon yields. Finally, a log-likelihood function based on a decomposition of the prediction errors is maximised to obtain the optimal parameter vector.

2.1 Continuous-Time Term Structure Modelling

The price at time t of a zero-coupon or pure discount bond maturing at time T may be expressed as[9]

$$P(t,T) = \exp\left[-R(t,T)T\right]. \tag{1}$$

[7] A function $F:\Re^n \to \Re$ is affine if there exists some coefficients $a \in \Re$ and $b \in \Re^n$ such that $F(X) = a + b^T X, \quad \forall X \in \Re^n$.

[8] See Duan and Simonato (1995), Babbs and Nowman (1999), Chen and Scott (2002), De Jong (1998), Geyer and Pichler (1999) and Lund (1997) for applications of the Kalman filter to term structure models.

[9] Coupon-bearing bonds may be interpreted as a portfolio of pure discount bonds of various face values and maturities.

Equation (1) may be rearranged to define the yield to maturity or continuously compounded yield as

$$R(t,T)=\frac{1}{T}\ln P\left(t,T\right). \tag{2}$$

The spot rate of instantaneous maturity or the short rate is then

$$r(t)=\lim_{T\to t}R\left(t,T\right). \tag{3}$$

Let the continuously compounded forward rate for the period $\left(T,T+t\right)$ be defined as

$$\frac{P(t,T+\tau)}{P(t,T)}=\exp\left[-f(t,T,\tau)\tau\right]=\frac{P\left(\ln P(t,T+\tau)-\ln P(t,T)\right)}{\tau} \tag{4}$$

It follows that

$$\lim_{\tau\to 0}\frac{\partial}{\partial T}P\left(t,T\right)=f(t,T) \tag{5}$$

which may be rewritten as

$$P\left(t,T\right)=\exp\left[-\left(\int_t^T f(t,s)ds\right)\right] \tag{6}$$

Hence, combining equations (1) and (6) reveals that the yield to maturity or the spot rate for maturity T may be construed as the integral of the forward rates over the remaining time to maturity of a pure discount bond. That is

$$R\left(t,T\right)=\frac{1}{T}\left(\int_t^T f(t,s)ds\right) \tag{7}$$

The dynamics of the short rate is assumed to follow a diffusion process described by the stochastic differential equation

$$dr\left(t\right)=\mu\left(r,t\right)dt+\sigma\left(r,t\right)dW\left(t\right) \tag{8}$$

where $\mu(r,t)$ represents the drift parameter, $\sigma(r,t)$ is the instantaneous volatility of the short rate and $W(t)$ represents a Brownian motion or Wiener process.

Using the assumption given by equation (8) and Ito's lemma[10]

[10] Subscripts indicate the specific derivative.

$$
\begin{aligned}
dP(r,t,T) &= \frac{\partial P}{\partial r}\partial r + \frac{\partial P}{\partial t}\partial t + \frac{\sigma^2}{2}\frac{\partial^2 P}{\partial r^2}\partial dt \\
&= \left(\mu P_r + P_t + \frac{\sigma^2}{2}\right)\partial t + \sigma P_r dW
\end{aligned}
\tag{9}
$$

Dividing by dt and taking expectations, as well as assuming that $E\left[dP/dt\right] = r(1+\nu)P$ yields

$$
\begin{aligned}
E\left(\frac{dP}{dt}\right) &= \mu P_r + P_t + \frac{\sigma^2}{2}P_{rr} \\
0 &= \mu P_r + P_t + \frac{\sigma^2}{2}P_{rr} - E\left(\frac{dP}{dt}\right) = \mu P_r + P_t + \frac{\sigma^2}{2}P_{rr} - r(1+\nu)P
\end{aligned}
\tag{10}
$$

where ν denotes the risk premium. Invoking the no-arbitrage condition and logarithmic investor preferences yield the Sharpe ratio

$$
\frac{E\left[R_i - r\right]}{\sigma_{R_i}} = \frac{\nu}{R_i} = \lambda \tag{11}
$$

Hence, the basic differential equation to be solved in this equilibrium pricing model is

$$
0 = \mu P_r + P_t + \frac{\sigma^2}{2}P_{rr} - +rP + \lambda r\sigma P_r \tag{12}
$$

where $\sigma_{R_P} = \dfrac{r\sigma}{P}P_r$ using Ito's lemma implies $\nu = \lambda r_{R_P} = \left(\lambda\sigma/P\right)P_r$.

2.1 Multifactor Affine Models

Multifactor affine models of the term structure represent the yields of securities as affine functions of a vector of K unobservable state variables or factors, $X = (X_1, X_2, \ldots, X_K)'$, which is governed by the following multidimensional diffusion process

$$
\underset{K\times 1}{d\,X(t)} = \mu\left[\underset{K\times 1}{X(t)}\right]dt + \sigma\left[\underset{K\times K}{X(t)}\right]\underset{K\times 1}{d\,W(t)} . \tag{13}
$$

The instantaneous short rate is given as

$$
r(t) = \beta_0 + \sum\nolimits_{i=1}^{K} \beta_i X_i(t) . \tag{14}
$$

The factors $X_i(t)$ are assumed to be independently generated by the Ornstein-Uhlenbeck (O-U) process in the Vasicek (Gaussian) case represented as

$$dX_i(t) = \kappa_i(\theta_i - X_i(t))dt + \sigma_i dW_i(t), \quad i = 1,\ldots,K \tag{15}$$

and the square-root process in the CIR (non-Gaussian) case represented as

$$dX_i(t) = \kappa_i(\theta_i - X_i(t))dt + \sigma_i \sqrt{X_i(t)}dW_i(t), \quad i = 1,\ldots,K \tag{16}$$

where κ_i, θ_i and σ_i are the speed of mean reversion, long-term mean and volatility parameters, respectively, and $W_i(t)$ denote independent Wiener processes under the risk-neutral pricing measure ,Φ .

The nominal pricing formula for a pure discount bond with a face value of $1 maturing at T is

$$P(T) = \prod_{i=1}^{K} A_i(T)\exp\left(-\sum_{i=1}^{K} B_i(T)X_i(t)\right) \tag{17}$$

where $B_i(T)$ and $A_i(T)$, in the Vasicek model have the following forms

$$B_i(T) = \frac{1}{\kappa_i}\left(1 - \exp(\kappa_i T)\right) \tag{18}$$

$$A_i(T) = \exp\left[\frac{\left(B_i(T) - T\right)\left(\kappa_i^2\left(\theta_i - \lambda_i\sigma_i / \kappa_i\right) - \sigma_i^2 / 2\right)}{\kappa_i^2} - \frac{\sigma_i^2 B_i(T)^2}{4\kappa_i}\right] \tag{19}$$

and where $A_i(T)$ and $B_i(T)$, in the CIR model have the following forms

$$A_i(T) = \left[\frac{2\gamma_i \exp\left[(\kappa_i + \lambda_i + \gamma_i)T/2\right]}{2\gamma_i \exp(\gamma_i T) + (\kappa_i + \lambda_i + \gamma_i)\left(1 - \exp(\gamma_i T)\right)}\right]^{\frac{2\kappa_i\theta_i}{\sigma_i^2}} \tag{20}$$

$$B_i(T) = \frac{2\left(1 - \exp(\gamma_i T)\right)}{2\gamma_i \exp(\gamma_i T) + (\kappa_i + \lambda_i + \gamma_i)\left(1 - \exp(\gamma_i T)\right)} \tag{21}$$

and $\gamma_i = \sqrt{(\kappa_i + \lambda_i)^2 + 2\sigma_i^2}$. The risk premium for each state variable is $\lambda_i X_i$ where the fixed parameter λ_i is the market price of risk for the corresponding state variable and is negatively related with the risk premium.

The pricing formula for a coupon bond with a face value of \$1 maturing at T with m coupons, C_i, to be paid at T_i is $\Psi(T)=\sum_{i=1}^{m} C_i P(T)$, with an implied yield to maturity obtained by solving $\Psi(T)=\sum_{i=1}^{m} C_i \exp(-\varphi T_i)$. However, $\varphi(X,T)$ would not be normally distributed given its nonlinear relationship with $X(t)$.

3.0 The State-Space Approach to Estimate Multi-Factor Term Structure Models

A state-space approach is adopted in this paper to estimate the unknown parameters and extract the unobservable state variables. A state-space representation is a dynamic system that comprises measurement equations, which condition observed variables on unobserved or state variables, as well as transition equations, which describe the path of the state variables. This system may be expressed in a form that may be examined using the Kalman filter which originates from the engineering control literature. The Kalman filter is an algorithm for sequentially updating a linear projection for the system using information from the observed variables.[11] The exact state-space representation for a multi-factor model with state vector $X(t)$ is based on the assumption that $X(0), X(1), \ldots, X(t)$ is a Markov process with $X(0) \sim \delta(0)[X(0)]$ and $X(t)|X(t-1) \sim \delta\left[X(t)|X(t-1)\right]$ where $\delta(0)[X(0)]$ and $\delta\left[X(t)|X(t-1)\right]$ represent the density of the initial state vector and the transition density, respectively.

3.1 The CIR (non-Gaussian) model

Consider the following CIR square-root process for the spot interest rate

$$dr(t)=\kappa\big(\theta-r(t)\big)dt+\sigma\sqrt{r(t)}dW(t) \tag{22}$$

Substituting equation (22) into equation (9) yields the following basic differential equation to be solved in the CIR model[12]

$$rP\left(1+\lambda P_r P\right)=\kappa(\theta-r)P_r+P_t+\frac{r\sigma^2}{2}P_{rr} \tag{23'}$$

[11] See Hamilton (1994).
[12] See Benninga and Wiener (1998).

which produces the following solution

$$P(T) = A(T)\exp\left(-B(T)r(t)\right) \tag{24}$$

where $A(T)$ and $B(T)$ are matrices with individual elements depicted by equations (20) and (21), respectively. The individual elements of $X(T)$ and $Y(T)$ are

$$X_i(t) = \theta_i\left(1-\exp\left(-\kappa_i\Delta t\right)\right) + \exp\left(-\kappa_i\Delta t\right)X_i(t-1) + \eta_i(t),$$

$$\eta(t)\big|\Omega(\text{t-1})\sim\text{N}\left(0,\Sigma(\text{t})\right);\ i = 1,\ldots,K \tag{25}$$

or, in the $K = 3$-factor case

$$\begin{bmatrix} X_1(t) \\ X_2(t) \\ X_3(t) \end{bmatrix} = \begin{bmatrix} \theta_1\left(1-\exp\left(-\kappa_1\Delta t\right)\right) \\ \theta_2\left(1-\exp\left(-\kappa_2\Delta t\right)\right) \\ \theta_3\left(1-\exp\left(-\kappa_3\Delta t\right)\right) \end{bmatrix} + \begin{bmatrix} \exp\left(-\kappa_1\Delta t\right) & 0 & 0 \\ 0 & \exp\left(-\kappa_2\Delta t\right) & 0 \\ 0 & 0 & \exp\left(-\kappa_3\Delta t\right) \end{bmatrix}\begin{bmatrix} X_1(t-1) \\ X_2(t-1) \\ X_3(t-1) \end{bmatrix} + \begin{bmatrix} \eta_1(t) \\ \eta_2(t) \\ \eta_3(t) \end{bmatrix}$$

and

$$Y_i\left(s_j\right) = \sum_{i=1}^{K}\frac{-\ln A_i\left(t,s_j\right)}{s_j-t} + \frac{B_i\left(t,s_j\right)X_i(t)}{s_j-t},\ j = 1,\ldots,M \tag{26}$$

The limit of the yield to maturity, or the long-term yield, as the time to maturity gets longer is $Y_i(\infty) = \lim_{T\to\infty} -\left(\log P(T)/T\right) = 2\kappa_i\theta_i/(\kappa_i+\lambda_i+\gamma_i)$.

The unobservable state variables for the CIR model are distributed conditionally as non-central χ^2 variates. In order to estimate the unobservable state variables, the exact transition density is substituted by a normal density $X(t)\big|X(t-1)\sim N\left(\mu(t),\Sigma(t)\right)$. The matrices for the conditional mean and conditional variance of $X(t)$ for the CIR model are determined such that they are equal to the first two moments of the exact transition density with elements defined as

$$\mu_i(t) = \theta_i\left[1-\exp\left(-\kappa_i\Delta t\right)\right] + \exp\left(-\kappa_i\Delta t\right)Y_i(t-1) \tag{27}$$

and the matrix, $\Sigma(t)$, has K diagonal elements

$$\Sigma_i(t)=\left[\frac{1-\exp(-\kappa_i\Delta t)}{\kappa_i}\right]\left[\frac{1}{2}\theta_i\sigma_i^2\left[1-\exp(-\kappa_i\Delta t)\right]+\exp(-\kappa_i\Delta t)Y_i(t-1)\right] \tag{28}$$

3.2 The Vasicek (Gaussian) model

Consider the following Vasicek spot interest rate *O-U* process

$$dr(t)=\kappa\big(\theta-r(t)\big)dt+\sigma dW(t) \tag{22'}$$

Hence, the basic differential equation after substituting equation (22') into equation (9) yields

$$0=\kappa(\theta-r)P_r+P_t+\frac{\sigma^2}{2}P_{rr}-+rP+\lambda r\sigma P_r \tag{29}$$

or

$$P(T)=\exp\left[\frac{1}{\kappa}\big(1-\exp(-\kappa T)\big)\big(Y(\infty)-r\big)-TY(\infty)-\frac{\sigma^2}{4\kappa^3}\big(1-\exp(-\kappa T)\big)^2\right] \tag{30}$$

which produces the following solution

$$P(T)=A(T)\exp\big(B(T)r(t)\big) \tag{24'}$$

where $B(T)$ and $A(T)$ are matrices with individual elements depicted by equations (18) and (19), respectively. The individual elements of $X(T)$ and $Y(T)$ are the same as equations (25) and (26) for the CIR model. The matrices for the conditional mean and conditional variance of $X(t)$ for the Vasicek model are

$$\mu_i(t)=\theta_i\left[1-\exp(-\kappa_i\Delta t)\right]+\exp(-\kappa_i\Delta t)Y_i(t-1) \tag{27'}$$

and

$$\Sigma_i(t)=\frac{\sigma_i^2}{2\kappa_i}\left[\frac{1-\exp(-2\kappa_i\Delta t)}{\kappa_i}\right] \tag{28'}$$ [13]

[13] See Dullmann and Windfuhr (2000).

3.3 The Kalman Filter

The continuously compounded yield to maturity on a pure discount bond is

$$Y_i(T) = \sum_{i=1}^{K} -\frac{\ln A_i(T)}{T} + \frac{B_i(T)X_i(t)}{T} \tag{31}$$

which affine in the unobserved vector of state variables $X_i(t)$. In order to estimate the system, it is assumed that yields for the N maturities are observed with errors of unknown magnitudes. Hence, equation (31) may be expressed as

$$Y_i(T) = \sum_{i=1}^{K} -\frac{\ln A_i(\beta;T)}{T} + \frac{B_i(\beta;T)X_i(t)}{T} + \varepsilon(t) \tag{32}$$

where $\beta = (\theta\ \kappa\ \sigma\ \lambda\ h)'$ is a vector of unknown parameters and $\varepsilon(t)$ has zero mean and variance, $H(t)$, but not necessarily normally distributed. Equation (32), which is the measurement equation of the state-space model, is expressed in stacked form as

$$\underset{N\times 1}{\begin{bmatrix} Y(X(t);\beta,T_1) \\ Y(X(t);\beta,T_2) \\ \vdots \\ Y(X(t);\beta,T_N) \end{bmatrix}} = \underset{N\times 1}{\begin{bmatrix} -\ln(A(\beta;T_1))/T_1 \\ -\ln(A(\beta;T_2))/T_2 \\ \vdots \\ -\ln(A(\beta;T_N))/T_N \end{bmatrix}} + \underset{N\times K}{\begin{bmatrix} (1/T_1)B(\beta;T_1) \\ (1/T_2)B(\beta;T_2) \\ \vdots \\ (1/T_N)B(\beta;T_N) \end{bmatrix}} \underset{K\times 1}{X(t)} + \underset{N\times 1}{\begin{bmatrix} \varepsilon_1(t) \\ \varepsilon_2(t) \\ \vdots \\ \varepsilon_N(t) \end{bmatrix}},$$

where $$\varepsilon(\mathrm{t}) \sim \mathrm{N}(0, H(t)) \tag{33}$$

$$H(t) = \begin{bmatrix} h_1(\mathrm{t}) & 0 & \cdots & 0 \\ 0 & h_2(\mathrm{t}) & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & h_N(t) \end{bmatrix}.$$

The transition equation of the state-space model over the time interval Δt of the discrete sample may be expressed as

$$X(t+1) = \Gamma(X(t);\beta,\Delta t) + \Sigma(X(t);\beta,\Delta t)^{1/2}\,\omega(t+1) \tag{34}$$

where $\Gamma(X(t);\beta,\Delta t) = E(X(t+\Delta t)|X(t))$, $\Sigma(X(t);\beta,\Delta t) = Var(X(t+\Delta t)|X(t))$ and $\omega(t+1)$ is a $K\times 1$ error vector with zero mean and unit variance.

The Kalman filter provides an optimal solution to predicting, updating and evaluating the likelihood function for Gaussian state-space models. For the non-Gaussian case, the Kalman filter may be used to extract approximate first and second moments of the model. In these models the Kalman filter is quasi-optimal and may be used to construct an approximate quasi-likelihood function.

Define the mean state matrix as

$$\Gamma\left(\hat{X}(t);\beta,\Delta t\right)=a(\beta,\Delta t)+b(\beta,\Delta t)\hat{X}(t) \tag{35}$$

and the state covariance matrix as

$$P(t+1|t)=Var\left(X(t+1)\middle|\Omega(t)\right) \text{ and } P(t)=Var\left(X(t)\middle|\Omega(t)\right) \tag{36}$$

where $X(t)=E\left(X(t)\middle|\Omega(t)\right)$, $\Omega(t)$ represents the information available at time t and $a(\cdot)$ and $b(\cdot)$ are $K\times 1$ and $K\times K$ matrices, respectively.

Equations (33) and (34) describe the state space representation. The Kalman filter provides optimal estimates, $\hat{X}(t+1)$, of the state variables given information at time $t+1$. The conditional mean and variance of $\hat{X}(t+1)$ may be expressed as

$$\hat{X}(t+1|t)=E(t)\left\{X(t+1)\right\}=a(\beta)+b(\beta)\hat{X}(t) \tag{37}$$

and

$$P(t+1|t)=E(t)\left\{\left[X(t+1)-\hat{X}(t+1|t)\right]\left[X(t+1)-\hat{X}(t+1|t)\right]'\right\} \tag{38}$$

Given $\Sigma\left(X(t);\beta,\Delta t\right)$ is affine in $X(t)$ and $Cov\left(X(t),\Sigma\left(X(t);\beta,\Delta t\right)^{1/2}\omega(t+1)\middle|\Omega(t)\right)=0$ and using the law of iterated expectations

$$P(t+1|t)=b(\beta,\Delta t)P(t)b(\beta,\Delta t)'+\Sigma\left(\hat{X}(t);\beta,\Delta t\right). \tag{39}$$

Equations (37) and (39) are referred to as the *prediction* step.

The second step in calculating the Kalman filter involves updating the estimation from the prediction step given the arrival of new information based on actual observations, $Y(t)$. Hence, the optimal estimates of the state vector and state covariance matrix are given by

$$\hat{X}(t+1) = \hat{X}(t+1|t) + K(t+1)\nu(t+1) \tag{40}$$

and

$$P(t+1) = P(t+1|t) - K(t+1)B(t+1)P(t+1|t) \tag{41}$$

where

$$\nu(t+1) = Y(t+1) - Y(t+1|t) \tag{42}$$

$$Y(t+1|t) = B(t)\hat{X}(t+1|t) + A(t) \tag{43}$$

$$K(t+1) = P(t+1|t)B(t+1)' F(t+1)^{-1} \tag{44}$$

$$F(t+1) = B(t+1)P(t+1|t)B(t+1)' + H(t+1) \tag{45}$$

Equations (39) and (40) are referred to as the *update* step and equations (42) to (45) are the observation estimation error, transition estimation, Kalman gain and covariance matrix of $R(t+1|t)$, respectively. For the Kalman filter to provide an optimal estimation of $\hat{X}(t+1)$, the following condition must hold

$$Cov\left[\left(X(t+1) - \hat{X}(t+1)\right), Y(s)\right] = 0; \quad s = 1,\ldots,t+1. \tag{46}$$

The log-likelihood function may be expressed as

$$\log L\left(Y(1),\ldots,Y(N);\beta\right) = -\frac{1}{2}\log\left[2\pi(T-1)N\right] - \frac{1}{2}\sum_{i=1}^{N}\log\left|F_i(t+1)\right| - \frac{1}{2}\sum_{i=1}^{N}\nu_i'(t+1)F_i^{-1}(t+1)\nu_i(t+1), \tag{47}$$

with the inverse and determinant of $F_i(t+1)$ expressed as

$$\begin{aligned} F_i^{-1}(t+1) &= H(t+1)^{-1} - H(t+1)^{-1}B(t+1)\left(P(t+1|t)^{-1} + B(t+1)'H(t+1)^{-1}B(t+1)\right)^{-1}B(t+1)'H(t+1)^{-1}, \\ \left|F_i(t+1)\right| &= \left|H(t+1)\right| * \left|P(t+1|t)\right| * \left|P(t+1|t)^{-1} + B(t+1)'H(t+1)^{-1}B(t+1)\right|. \end{aligned} \tag{48}$$

In the Gaussian case, the conditional mean and variance of the system is correctly specified. Thus, the measurement and transition equations and the Kalman filter recursion can be used to conduct prediction-error decomposition in the evaluation of the exact likelihood function. However, in the non-Gaussian case, the linear Kalman filter does not produce $\hat{X}(t+1)$ but rather $\bar{X}(t+1)$, the linear projection of $X(t+1)$ on the linear sub-space generated by the observed yields. This linearly optimal approximation yields a quasi-likelihood function. As discussed in Bollerslev and Wooldridge (1992), the hyperparameter vector $\hat{\beta}(T)$ that maximises the quasi-likelihood function in the non-Gaussian case is approximately consistent and asymptotically normal. Alternatively, in the Gaussian case, the quasi-likelihood function turns into the exact likelihood function given normally distributed measurement errors. The asymptotic distribution of $\beta = (\theta\ \kappa\ \sigma\ \lambda\ h)'$ is

$$\sqrt{T}\left(\hat{\beta}(T)-\beta(0)\right) \sim N\left(0, \hat{F}(T)^{-1}\hat{G}(T)\hat{F}(T)^{-1}\right) \tag{47}$$

where

$$\hat{F}(T) = \frac{1}{T}\sum_{t=1}^{T} f\left(\hat{\beta}(T);Y(t),T\right) \tag{48}$$

$$\hat{G}(T) = \frac{1}{T}\sum_{t=1}^{T} \frac{\partial \ln l\left(\hat{\beta}(T);Y(t),t\right)'}{\partial\hat{\beta}} \frac{\partial \ln l\left(\hat{\beta}(T);Y(t),t\right)}{\partial\hat{\beta}} \tag{49}$$

and

$$f\left(\hat{\beta};Y(t),t\right) = \frac{\partial\psi(t)'}{\partial\beta}\Psi(t)^{-1}\frac{\partial\psi(t)}{\partial\beta} + \frac{1}{2}\frac{\partial\Psi(t)'}{\partial\beta}\left(\Psi(t)^{-1}\otimes\Psi(t)^{-1}\right)\frac{\partial\Psi(t)}{\partial\beta} \tag{50}$$

$$L\left(\beta;Y(T),T\right) = \sum_{t=1}^{T} l\left(\beta;Y(t),T\right) \tag{51}$$

where ψ and Ψ are the conditional mean and variance functions from the linear Kalman filter.[14]

4.0 Estimation Results of Multi-factor Models

4.1 Data Description

The data used in the empirical study consists of daily zero coupon GOJ domestic bond yields from 24 September 2004 to 28 July 2006 obtained from Bloomberg. In particular, the panel data set covers 435

[14] See Duan and Simonato (1998).

observations and N=15 interest rates. The maturities included 0.25-, 0.5-, 1-, 2-, 3-, 4-, 5-, 6-, 7-, 8-, 9-, 10-, 15-, 20- and 30-year tenors.

Table 1. Summary Statistics: GOJ Zero Coupon Bond Yields 9/24/2004 - 7/28/2006

Maturity	3 mth	6 mth	1 yr	2 yr	3 yr	4 yr	5 yr	6 yr	7 yr	8 yr	9 yr	10 yr	15 yr	20 yr	30 yr
Mean	14.75	14.84	14.97	15.27	15.52	15.77	16.05	16.36	16.69	16.87	17.01	17.11	17.76	19.28	30.90
Median	13.96	14.31	14.35	14.86	15.17	15.36	15.56	15.77	16.10	16.51	16.66	16.79	17.21	19.23	30.61
Maximum	17.32	17.32	17.32	17.47	17.77	18.03	18.64	19.35	20.14	20.43	20.38	20.37	20.63	21.65	70.27
Minimum	13.22	13.29	13.30	13.44	13.52	13.66	13.80	13.98	14.06	14.14	14.24	14.33	14.92	15.98	18.49
Std. Dev.	1.37	1.32	1.26	1.24	1.31	1.38	1.48	1.59	1.72	1.71	1.63	1.58	1.44	1.46	8.16
Skewness	0.72	0.71	0.67	0.60	0.58	0.56	0.55	0.59	0.64	0.67	0.63	0.60	0.54	-0.13	0.85
Kurtosis	1.93	1.96	1.93	1.93	1.88	1.85	1.86	1.93	2.06	2.22	2.20	2.19	2.14	1.95	4.82

The average yield per maturity over the sample period indicates that, on average, the GOJ zero coupon term-structure is upward sloping (see Table 1). The average spread or premium between the 3-month and 30-year spot rates is approximately 1 500 basis points. This significant risk premium of 8.0 per cent demanded by investors is likely to be caused by unfavourable GOJ debt ratios. The volatility of the spot rates is greatest at the 30-year maturity and lowest at the 3-month to 4-year maturities. This is inconsistent with expectations of greater volatility at the shorter maturities which may be due to greater uncertainty regarding the riskiness of GOJ bonds. The skewness and kurtosis parameters indicate that the distributions are not normal across maturities.[15] The skewness coefficient of all yields, except the 20-year yield, is greater than zero indicating a lower downside risk relative to the normal distribution. The kurtosis values below 3 for all yields apart from the 30-year maturity, implies lower losses when compared to the normal distribution.

The zero-coupon yields on the GOJ bonds are highly correlated (>80.0 per cent) across all maturities, abstracting from the 20- and 30-year maturities which exhibit much lower correlation coefficients (see Table 2). For the most part, the correlations are close to perfect between yields on maturities up to one year apart. As the number of years increase between maturities, these pair-wise correlation coefficients decline, suggesting the use of a multi-factor term structure model.

[15] The skewness and kurtosis of the Normal distribution is 0 and 3, respectively.

Table 2. Correlation Matrix: GOJ Zero Coupon Bond Yields 9/24/2004 - 7/28/2006

	3-month	6-month	1-year	2-year	3-year	4-year	5-year	6-year	7-year	8-year	9-year	10-year	15-year	20-year	30-year
3-month	1.00	0.98	0.96	0.93	0.92	0.92	0.93	0.93	0.93	0.91	0.90	0.90	0.92	0.81	0.48
6-month		1.00	0.99	0.95	0.94	0.94	0.95	0.94	0.94	0.92	0.91	0.91	0.93	0.83	0.48
1-year			1.00	0.99	0.98	0.98	0.98	0.97	0.96	0.95	0.94	0.94	0.95	0.83	0.46
2-year				1.00	1.00	1.00	0.99	0.98	0.97	0.97	0.96	0.96	0.96	0.79	0.39
3-year					1.00	1.00	0.99	0.98	0.97	0.97	0.96	0.96	0.96	0.79	0.38
4-year						1.00	1.00	0.99	0.98	0.98	0.97	0.97	0.97	0.81	0.41
5-year							1.00	1.00	0.99	0.99	0.98	0.98	0.98	0.82	0.43
6-year								1.00	1.00	0.99	0.99	0.99	0.99	0.83	0.45
7-year									1.00	1.00	0.99	0.99	0.99	0.83	0.46
8-year										1.00	1.00	1.00	0.99	0.80	0.43
9-year											1.00	1.00	0.98	0.78	0.41
10-year												1.00	0.98	0.79	0.41
15-year													1.00	0.89	0.52
20-year														1.00	0.71
30-year															1.00

4.2 Empirical Results

One-, two-, and three factor Vasicek and CIR models are estimated to obtain the parameters estimates of λ, κ, θ and σ, the standard deviation estimates of the N measurement errors, $\sqrt{h_i}$, as well as the values for the log-likelihood and Akaike Information Criterion (AIC)[16] (see Tables 3 and 4; standard errors are shown in italics).

The results for the Vasicek model indicate that all of the λ, θ and σ parameters are statistically insignificant at the 5.0 per cent level (see Table 3). In addition, the standard errors are generally very large and in most cases increase significantly as the number of factors increases. The results are mixed for the κ parameters. The κ parameters are statistically significant in the two- and three-factor models but not significant in the one-factor model. All of the estimated standard deviation parameters for the measurement errors are statistically significant. The log-likelihood values show strong increases as the number of factors increase. However, only one of the 15 estimated standard deviation parameters for the measurement errors displays a consistent decline as the number of factors increase. The smallest standard deviations for measurement equation in the Vasicek models are 2, 3 and 0 basis points for the 5-year bond rate in the one-, two- and three-factor models, respectively. The largest standard deviations are 1 333, 1285 and 1195 basis points for the 30-year bond rate in the one-, two- and three-factor models, respectively. These large measurement errors suggest that the models are unable to explain a significant portion of the 30-year yield movements.

[16] The initial starting values chosen for these parameters were the same across both models. Further, the parameter estimates were robust to variations in the starting values.

The parameter results from the CIR model estimation produced significantly more favourable results (see Table 4). Most of the λ, κ, θ and σ parameter estimates are statistically significant at the 5 percent level, except θ_1 in the one-factor model and λ_1, θ_1, θ_2, κ_1 and σ_2 in the three-factor model. All of the parameter estimates are statistically significant for the two-factor model. The estimates of the market price of risk parameter, λ, for the CIR models have plausible values. These estimated parameters also have large negative values, indicating the existence of large and positive risk premia for the latent factors.[17]

The estimates of the rate of mean reversion parameter, κ, are also significant except for the first factor of the three-factor model. These estimates range from 0.5 to 0.8, indicating that the mean half lives, or the expected time for the short rate to return halfway to its long-term average mean, ranges between 0.9 to 1.4 years.[18] This narrow range of mean half-life values implies that mean reversion for GOJ rates is relatively fast and that the factor determines variations primarily at the short end of the yield curve. The values for the volatility estimates, σ, are statistically significant and small (12 basis points for each factor), indicating a relatively smooth process of mean reversion. Half of the parameter estimates for long-term average rate, θ, are significant. However, their values are very close to zero and the condition $2\kappa\theta < \sigma^2$ does not hold, indicating that zero could be a reflecting barrier for the process. The correlation coefficient between factors one and two in the two factor model is -0.99. The log-likelihood value and AIC values improve by 2.1 per cent when moving from the one-factor model to the two-factor model but deteriorates notably (-7.6 per cent) when moving to the three-factor model.[19] This is taken as evidence that the two-factor model out-performs the one- and three-factor models.

[17] Some examples of risk premium estimates for the 'level' factor using CIR models in the literature include: -0.1 and 0.0 for the UK and German term structure over 6/1/99 – 28/1/04, respectively (see Chatterjee (2005); -0.2 and 1.1 for the US two-factor and three-factor term structure models over 1/83 – 12/88 (see Chen and Scott (2002).

[18] The half life is computed using: $\exp(-\kappa_j t) \Rightarrow t = -\ln(0.5)/\kappa_j$.

[19] The likelihood ratio (LR) statistic rejects the null hypotheses that the additional factors are not jointly significant at the 1.0 per cent level. However the LR test is unreliable in this case because it does not have the standard asymptotic χ2 distribution when the errors are not Gaussian.

Table 3.
Estimates from Vasicek Model for GOJ Bond Yields: 9/24/2004 to 7/28/2006

	One Factor Model	Two Factor Model	Three Factor Model
λ_1	-0.5221	-1.9994	-0.8451
	(91.078)	*(519.8486)*	*(22665.79)*
λ_2		-0.1531	-0.5714
		(452.6597)	*(876.8248)*
λ_3			-11.5331
			(23186.01)
θ_1	-0.4163	-0.1728	0.0241
	(421.6795)	*(66.5715)*	*(4001.42)*
θ_2		0.1581	-0.0239
		(250.7464)	*(1186.39)*
θ_3			0.0247
			(22.7575)
κ_1	0.0054	0.2512	0.3544
	(0.0029)	*(0.0163)*	*(0.0225)*
κ_2		0.0246	0.0663
		(0.0052)	*(0.0079)*
κ_3			1.1638
			(0.0670)
σ_1	0.0002	0.0010	0.0037
	(0.0527)	*(0.4045)*	*(2.0927)*
σ_2		0.0003	0.0000
		(0.2227)	*(1.3786)*
σ_3			0.0030
			(4.7800)
$\sqrt{h_1}$	0.0025	0.0060	0.0058
	0.0001	*0.0004*	*0.0003*
$\sqrt{h_2}$	0.0015	0.0045	0.0051
	0.0001	*0.0004*	*0.0004*
$\sqrt{h_3}$	0.0011	0.0025	0.0041
	0.0000	*0.0004*	*0.0007*
$\sqrt{h_4}$	0.0010	0.0007	0.0032
	0.0001	*0.0001*	*0.0006*
$\sqrt{h_5}$	0.0006	0.0010	0.0026
	0.0001	*0.0002*	*0.0015*
$\sqrt{h_6}$	0.0004	0.0008	0.0015
	0.0001	*0.0002*	*0.0011*
$\sqrt{h_7}$	0.0002	0.0003	0.0000
	0.0001	*0.0001*	*0.0000*
$\sqrt{h_8}$	0.0009	0.0012	0.0017
	0.0003	*0.0005*	*0.0010*
$\sqrt{h_9}$	0.0022	0.0027	0.0036
	0.0007	*0.0010*	*0.0021*
$\sqrt{h_{10}}$	0.0021	0.0024	0.0036
	0.0005	*0.0006*	*0.0017*
$\sqrt{h_{11}}$	0.0023	0.0023	0.0033
	0.0005	*0.0005*	*0.0011*
$\sqrt{h_{12}}$	0.0027	0.0024	0.0033
	0.0005	*0.0004*	*0.0006*
$\sqrt{h_{13}}$	0.0046	0.0052	0.0069
	0.0011	*0.0013*	*0.0016*
$\sqrt{h_{14}}$	0.0091	0.0098	0.0089
	0.0011	*0.0014*	*0.0009*
$\sqrt{h_{15}}$	0.1333	0.1285	0.1195
	0.0059	*0.0052*	*0.0041*
LogL	26 319.33	28 331.25	29 712.18
AIC	-120.9164	-130.1437	-136.4698

Table 4.
Estimates from CIR Model for GOJ Bond Yields: 9/24/2004 to 7/28/2006

	One Factor Model	Two Factor Model	Three Factor Model
λ_1	-0.8177	-0.7042	-0.0969
	(0.0026)	*(0.0145)*	*(0.4332)*
λ_2		-0.8206	-0.4453
		(0.0074)	*(0.0344)*
λ_3			-0.5964
			(0.0460)
θ_1	0.0000	-0.0002	-0.0015
	(0.0000)	*(<0.0001)*	*(0.0032)*
θ_2		0.0002	-0.0002
		(<0.0001)	*(0.0004)*
θ_3			0.0022
			(0.0006)
κ_1	0.7810	0.4859	0.2025
	(0.0013)	*(0.0180)*	*(0.4355)*
κ_2		0.5861	0.5156
		(0.0161)	*(0.0419)*
κ_3			0.6666
			(0.0557)
σ_1	0.0017	0.0012	0.0001
	(0.0003)	*(<0.0001)*	*(<0.0001)*
σ_2		0.0012	0.0006
		(<0.0001)	*(0.0003)*
σ_3			0.0005
			(0.0001)
$\sqrt{h_1}$	0.0050	0.0068	0.0373
	0.0003	*0.0010*	*0.0335*
$\sqrt{h_2}$	0.0043	0.0023	0.0171
	0.0003	*0.0001*	*0.0123*
$\sqrt{h_3}$	0.0030	0.0051	0.0061
	0.0003	*0.0013*	*0.0026*
$\sqrt{h_4}$	0.0023	0.0027	0.0023
	-0.0004	*0.0007*	*0.0007*
$\sqrt{h_5}$	0.0020	0.0026	0.0005
	0.0007	*0.0008*	*-0.0003*
$\sqrt{h_6}$	0.0012	0.0015	0.0003
	0.0004	*0.0002*	*0.0001*
$\sqrt{h_7}$	0.0006	0.0012	0.0001
	0.0001	*0.0002*	*0.0001*
$\sqrt{h_8}$	0.0332	0.0012	0.0007
	0.0228	*0.0003*	*0.0002*
$\sqrt{h_9}$	0.0039	0.0016	0.0015
	0.0018	*0.0004*	*0.0005*
$\sqrt{h_{10}}$	0.0038	0.0015	0.0012
	0.0017	*0.0002*	*0.0001*
$\sqrt{h_{11}}$	0.0034	0.0016	0.0020
	0.0012	*0.0004*	*0.0002*
$\sqrt{h_{12}}$	0.0033	0.0010	0.1165
	0.0009	*0.0002*	*0.0669*
$\sqrt{h_{13}}$	0.0056	0.0086	0.0138
	0.0016	*0.0018*	*0.0100*
$\sqrt{h_{14}}$	0.0106	0.0178	0.0241
	0.0015	*0.0028*	*0.0109*
$\sqrt{h_{15}}$	0.1437	0.1450	0.1325
	0.0072	*0.0120*	*0.0064*
LogL	25 531.71	26 068.27	24 076.15
AIC	-117.2952	-119.7392	-110.557

Two of the 15 estimated standard deviation parameters for the measurement errors tend to zero as the number of factors increase. The smallest standard deviations for measurement equation in the CIR models are 6 and 1 basis points for the 5-year bond rate in the one- and three-factor models, respectively, and 10 basis points for the 10-year bond in the two-factor model. The largest standard deviations are 1 437, 1 450 and 1 325 basis points for the 30-year bond rate in the one-, two- and three-factor models, respectively. Similar to the Vasicek models, these values are significantly larger compared to the relatively low standard deviations for the remaining bond rates. Hence, aside from the 30-year yield, the factors explain most of the yield fluctuations in the CIR models suggesting that the 30-year yield fluctuation is not adequately explained by the CIR model.

The time series evolution of the combined factors of the two-factor CIR model are compared with the evolutions of the 3-month to 20-year bond yields (see Figure 1). The combined factors are strongly correlated with these yields suggesting that monetary policy influences these yields. The correlation coefficients between the combined factors of the two-factor CIR model and GOJ yields range from 94.0 per cent to 100.0 per cent for the 3-month to 15-year yields and 82.0 per cent for the 20-year yield. The correlation between the combined factors and the 30-year yield was significantly lower with a value of 44.0 per cent (see Figure 2). The Kalman filter one-step ahead predicted yields and the actual yields for the two-factor CIR model are illustrated in Figure 3. The Kalman filter algorithm display accurate forecasting ability, particularly for the 4-year to 10-year GOJ maturity yields.

Figure 1. Evolution of Combined Factors of 2-Factor CIR Model and the 3-month to 20-year maturities

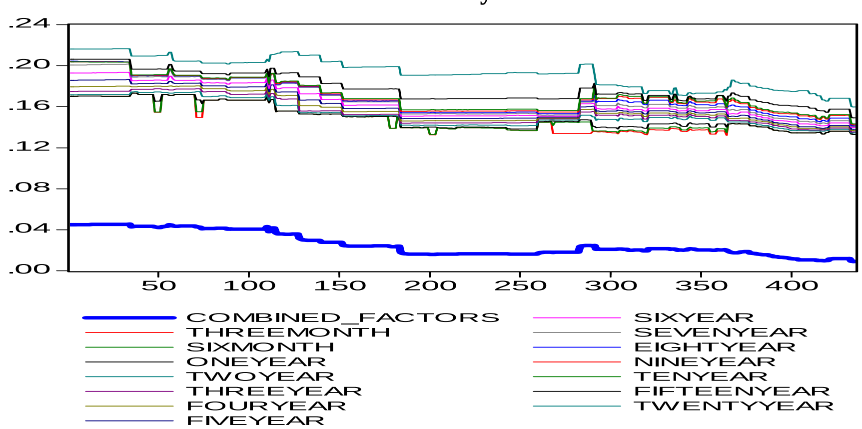
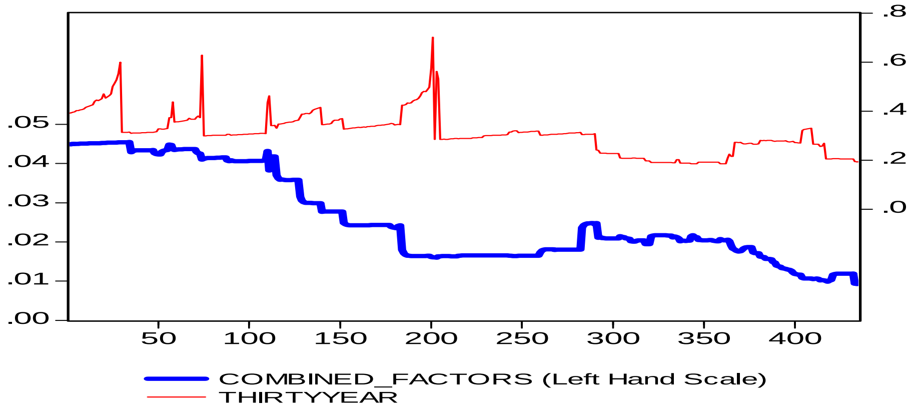


Figure 2. Evolution of Combined Factors of 2-Factor CIR Model and the 30-year maturity



4.3 Factor Loadings

The factor loadings as a function of maturity presented in this section is based on the estimated parameters of the measurement equation in the one- and two-factor CIR models (see Figures 4 and 5). The factor loadings are derived using the coefficients of $B(T)$ as expressed in equation (26). The term structure of zero yields can be one of three possible shapes. If the short rate, r, is less than $Y(\infty)$, then shape is monotonically increasing. It is monotonically decreasing or 'humped' when $r > Y(\infty)$.

As given by equation (14) the sum of all factors in a multi-factor term structure model is equal to the level of the instantaneous short rate. The coefficients on the factor of the one-factor model and the 1st factor of the two-factor model display the same pattern of rapid decline as the time to maturity increases, indicating a strong impact for the short-term rates. Specifically, these factor loadings display steep declines between 0 and 2.5 years. The declines become less steep as the time to maturity increases to around 20 years and level off at very low levels for the remaining maturities. These factors could represent 'level' factors.[20] The 2nd factor loading of the two-factor model exhibits a steep increase for short-term rates between 0 and 5 years which diminishes as the time to maturity increases to around 20 years and levels off for the remaining maturities. This factor could represent the 'steepness' factor corresponding to the slope of the yield curve.

[20] See Litterman and Scheinkman (1991).

Figure 3. Actual and Predicted GOJ Yields

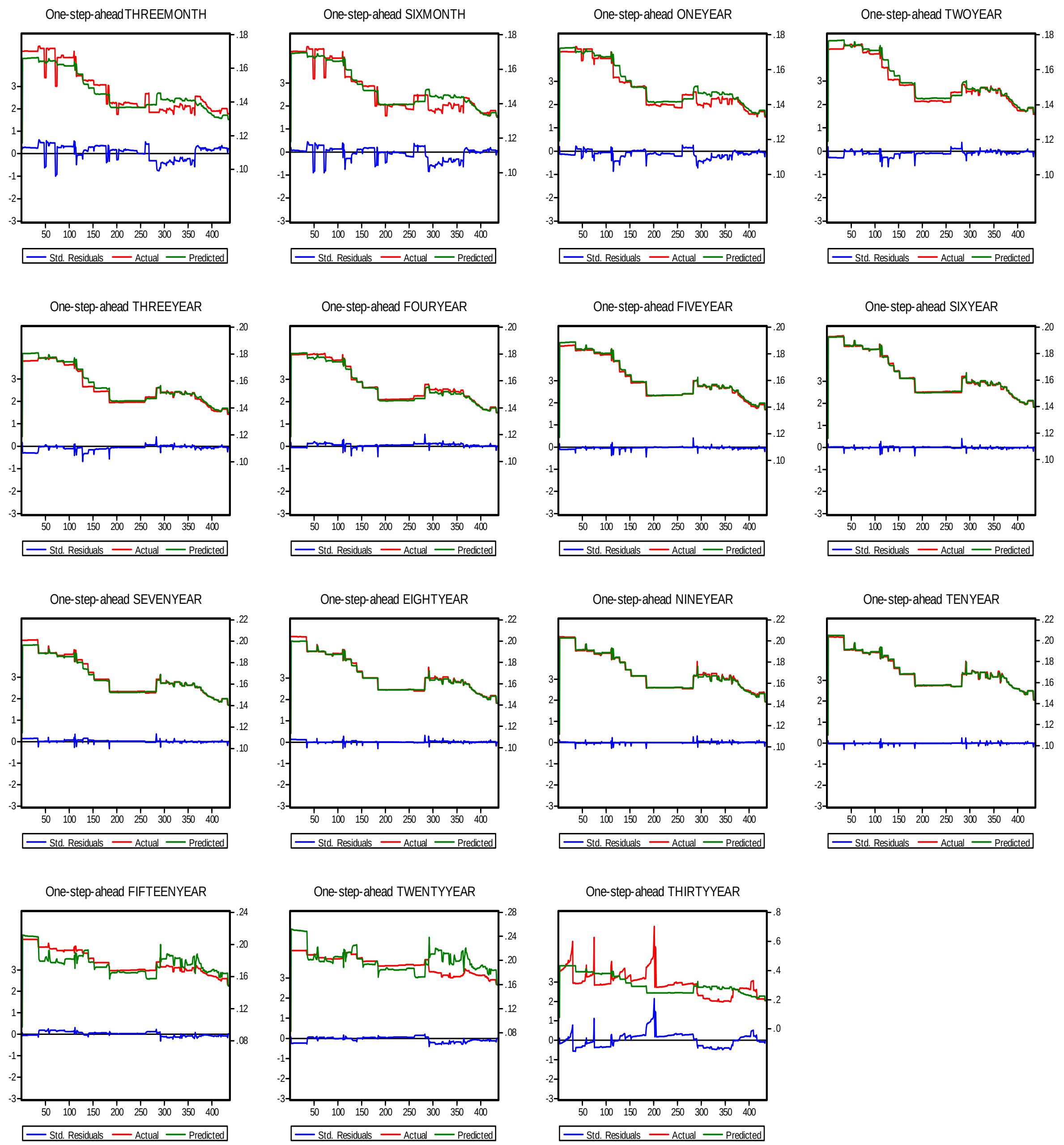


Figure 4. Factor Loading of One-Factor CIR Model

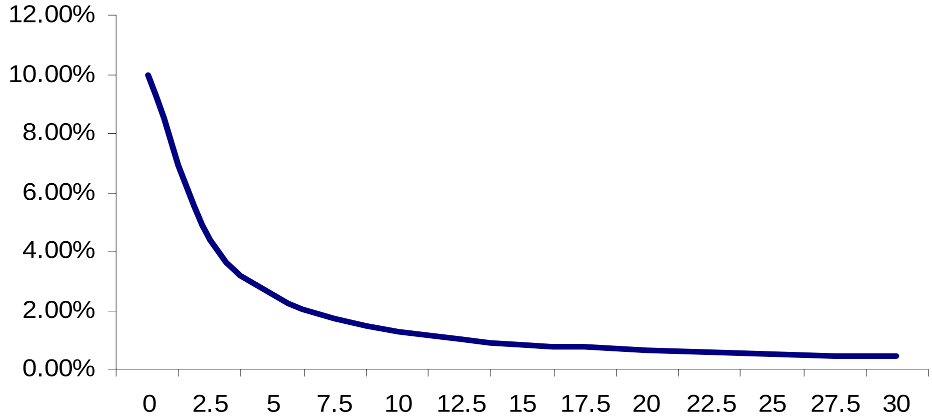
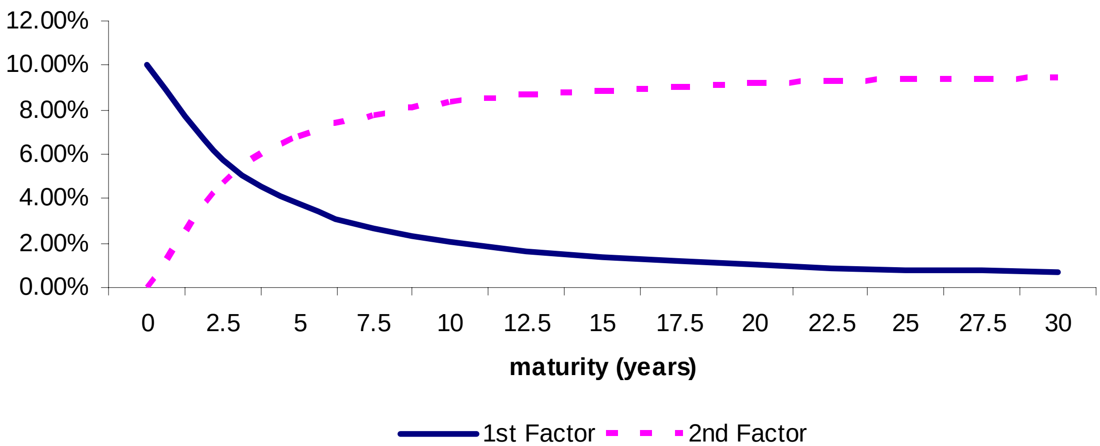


Figure 5. Factor Loadings of Two-Factor CIR Model



5.0 Conclusion

In this paper single- and multi-factor version of the Vasicek and CIR models of the term structure of interest rates were estimated using a state space formulation. This approach combines both cross-section and time series information based on a system of bond price equations to generate estimates of unobservable state variables that drive the term structure. The models are estimated for up to three factors using a quasi-maximum-likelihood estimator with a Kalman filter. Fifteen bond maturities were used comprising the 0.25-, 0.5-, 1-, 2-, 3-, 4-, 5-, 6-, 7-, 8-, 9-, 10-, 15-, 20- and 30-year computed zero-coupon GOJ bond yields covering the period 24 September 2004 to 28 July 2006 to estimate the parameters of each model.

Based on the empirical results, the Vasicek models performed very poorly relative to the CIR models. Additionally, the results suggested that the 2-factor CIR model provided the best representation of the dynamics of the yield curve. Based on the factor loadings, extracted factors of the two-factor model correspond with the general level and slope of interest rates, respectively. The empirical analysis for the 2-factor model revealed that the level of the short rate exhibited strong and smooth mean reversion and indicated the existence of a large and significant risk premium that increases with time to maturity. The values of the parameter estimates for the long-term average rate are all virtually zero. However, this is probably a result of the sample period under analysis. That is, the period corresponds to a consistent series of downward adjustments to Bank of Jamaica repurchase rates following a substantial upward adjustment of over 15 000 basis points during an episode of substantial foreign exchange market instability in 2003. The strong reversal of the short rate since 2003 could explain the dominant expectations of investors for considerable loosening of monetary policy being reflected in the estimated long-term average rate.

A summary of the key findings of this study, based on significant estimates from two-factor CIR model, are:

- The short-rate (influenced by monetary policy) exhibits rapid decline between 0 and 2.5 years which become less steep as the time to maturity increases to around 20 years and levels off to a very low level for the remaining bond maturities
- Risk premium parameters have large negative values, indicating the existence of a large and positive risk premia for the 'level and 'steepness' factors that increases with the time to maturity of GOJ bonds
- Long-run average yield parameters reveal that investors were expecting lower interest rates over the sample period
- Mean reversions for the 'level' and 'steepness' factors that drive the dynamics of GOJ yields are relatively fast and smooth indicating relatively short-lives for monetary shocks
- The 'level' and 'steepness' factors explain variations primarily at the short end of the yield curve
- The Kalman filter algorithm display accurate forecasting ability, particularly for the 4-year to 10-year GOJ maturity yields.

Similar to traditional research on the term structure, this study examined a 'yields-only' latent-factor model of the dynamics of the yield curve. Recent studies in the literature have focused on uncovering the relationship between term structure models and specific macroeconomic variables. Future research will explicitly incorporate the relationship between term structure latent factors and macroeconomic variables of interest in the Jamaica case. For example, based on estimated factor loadings, this study concluded that the unobserved short rate (related to the BOJ policy rate) has a significant impact on the short end of the yield curve and a relatively minimal impact on the long end. Relevant observable macroeconomic variables that could be jointly incorporated with latent state variables in a state-space model of the term structure include monetary aggregates, the expected inflation gap, the expected output gap, foreign interest rates, as well as the fiscal deficit to account for yield movements at the long end. [21]

Policy Recommendations

Modelling the term structure of interest rates (or the yield curve) is critical to explaining the behaviour of interest rates. The information derived from this process play vital roles in both policy formulation and financial market activities. Specifically, as mentioned in the introduction, understanding interest rate dynamics is important to monetary and debt management policies, as well as, the pricing of interest rate derivatives. The policy recommendations provided in this paper, however, are related primarily to monetary policy.

The study finds evidence that supports the usefulness of term structure models for monetary policy. Presently, the Bank of Jamaica infers market perception about the future path of interest rates by using information from the movements in monetary aggregates, infrequent auctions of mainly short-maturity bonds, as well as market intelligence via informal surveys. However, the estimation of interest rate term structure models would provide a more useful guide for the conduct of monetary policy. For instance, the slope of the term structure can be used for measurement of expected inflation and the prediction of GDP growth. That is, the slope of the yield curve reflects investors' perception with regard to the future path of short-term interest rates.

Results from this study reveal that the Kalman filter algorithm provides accurate forecasts of investor expectations regarding future short rates. The Bank can also influence the slope of the yield curve by changing the level of the short rate relative to the long rate. If the slope is positive and steep, this implies that inflation is expected to accelerate and *vice versa*. Of course, estimates on the "market price of risk" obtained from term structure models will allow the monetary authorities to purge the slope estimate of the

[21] See, for example, Rudebusch and Wu (2003) for an application of a 'macro-finance' term structure model to US Treasury yields.

GOJ credit risk and term premia. Future business cycles may also be predicted by the Bank given information provided by the slope estimates of term structure models. For example, a wealth of evidence exists in the literatures which find that yield curves accurately forecast the turning points of business cycles. In particular, negatively-sloped yield curves usually occur around business cycle peaks.[22]

The estimation results from the state-space model of GOJ term strucure should also be incorporated in the macro-econometric model of the Jamaican economy employed by the Bank to examine the monetary transmission mechanism. Specifically, including equations depicting interest rate term-structure relationships should improve the accuracy of inflation forecasts and, hence, the efficacy of monetary policy. The addition of expectations model-based term structure equations to this model would be specified to ensure that the five-year rate, for example, would move to equate with a series of annual investments at the expected one-year rate over the five years plus the estimated premia.

Finally, the Bank (as a GOJ bond issue agent) should partner with the GOJ to establish an accurate and reliable benchmark yield curve through the regular issue of GOJ bonds at *standardized* maturities along the entire yield curve. This will greatly benefit liquidity conditions as well as the price discovery process, for improved monetary and fiscal policy formulation, as well as more accurate pricing of risky bonds and hedging instruments at varying maturities. The Bank has already begun to assist in the development of a benchmark yield curve given current efforts to establish a central securities depository within a modern and efficient payment and securities settlement infrastructure.

[22] See Estrella and Trubin (2006).

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